MAC-CPTM Situations Project

Situation 24: Isn't Absolute Value Always Positive?

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Prompt

In a first-year algebra course, a discussion centered on solving absolute value inequalities. One student said that since everything in $|x + 3| < 5$ is positive, the solution could be found by solving $x + 3 < 5$, so the solution was $x < 2$.

Commentary

The primary issue here is that of understanding the nature of absolute value, specifically as it relates to inequalities. The goal is not simply to know the steps and methods for solving absolute value inequalities, but to have a deeper knowledge about the reasons why certain solutions are valid and others are not. There are several definitions of absolute value, some of which will be addressed in the following foci (specifically, Foci 2-4). Each definition adds a new way to think about absolute value, and to solve absolute value inequalities. Focus 1 emphasizes the seemingly obvious, but sometimes neglected, requirement that a solution set include all possible solutions and exclude all values that are not solutions. Finally, Focus 5 provides a way to think about absolute value inequalities graphically.

Mathematical Foci

Mathematical Focus 1

The solution statement must include all values which are solutions to $|x + 3| < 5$*, and exclude all other values.*

When solving an equation or inequality, the goal is to identify a set of values that includes *all* the solutions, and excludes any value that is *not* a solution.

The solution set, $x < 2$, contains numbers that are solutions to the given inequality and numbers that are not solutions to the given inequality. For example, $x = 1$ is a solution because $|1 + 3| < 5$ is a true statement. But consider x

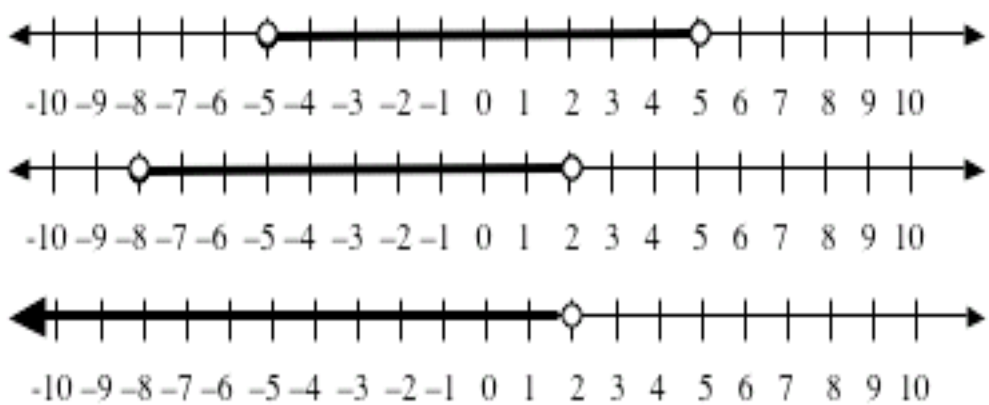
= -10 as a solution. Substituting this value into the original inequality yields the statement $|-10 + 3| < 5$, i.e. $|-7| < 5$, which is false. Exploring other values could ultimately lead to a determination of the smallest and largest solutions to $|x + 3| = 5$. Solutions to this equation are the boundaries on the solutions to $|x + 3| < 5$. All numbers between -8 and 2 are solutions to this inequality.

Mathematical Focus 2

Defining absolute value as distance from zero on a number line gives a clear picture of the meaning of $|x + 3| = 5$*, and allows* $|x + 3| < 5$ *to be solved algebraically.*

One definition of absolute value of real numbers involves distance from zero on the number line. Real-number solutions to inequalities can be expressed as graphs using number lines. These graphs may be expressed in terms of the value of the variable (e.g., x) that produce the solution, or in terms of the values of a variable expression (e.g., $x + a$) that conveys the solution. In these cases, graphs for $x + a$ are translations of the graphs for x.

One could interpret the question posed in this prompt as asking for all numbers (written $x + 3$) that are less than five units from zero. Graphing the set of numbers, x, that are less than five units from zero yields the first graph shown in the figure. To compensate for the "+3" and thus have a graph of the values of x that satisfy the inequality $|x + 3| < 5$ requires translating the first graph three units to the left, as shown in the second graph in the figure. A graph of $x < 2$ (the solution set of $x + 3 < 5$) is shown as the third graph in the figure. Comparing the second and third graphs illustrates that some solutions of $x + 3 < 5$ are not solutions of $|x + 3| < 5$.



Using this definition of absolute value (distance from zero on the number line) allows one to solve absolute value inequalities algebraically. When an inequality involves a numerical constraint on the absolute value of an algebraic expression in one variable, distance from zero can be used to write an extended inequality that can then be solved algebraically. One could interpret the question as asking for all numbers (written $x + 3$) that were at most five units from zero, thus generating the inequality $-5 < x + 3 < 5$ which can be solved algebraically to find the solution set $-8 < x < 2$. This solution set is more limited than the solution set for $x + 3 < 5$ (which is $x < 2$); that is, all x such that $x \leq -8$ are solutions to $x + 3 < 5$, but not to $|x + 3| < 5$.

Mathematical Focus 3

When $|x + 3|$ is defined using a piecewise function, absolute value inequalities can be solved algebraically.

The definition of absolute value as a piecewise function is: $|x| = \begin{cases} x, \text{if } x \geq 0 \\ -x, \text{if } x < 0 \end{cases}$. We can then define the left-hand member of the inequality as $|x + 3| = \begin{cases} x + 3, \text{if } x + 3 \geq 0 \\ -x - 3, \text{if } x + 3 < 0 \end{cases}$. The solution set of the given inequality is the union of the solution sets of the inequalities generated by these two pieces. Solving the system of inequalities suggested by the first piece, $x + 3 < 5$ and $x + 3 \geq 0$, gives us $-3 \leq x < 2$. Solving the system of inequalities suggested by the second piece, $-x - 3 < 5$ and $x + 3 < 0$, yields $-8 < x < -3$. The union of these two solution sets yields $-8 < x < 2$.

Mathematical Focus 4

Absolute value of a real number defined as the positive square root of the square of the number provides another way of understanding absolute value, and another tool for solving $|x + 3| < 5$.

Yet another definition of absolute value of a real number indicates the positive square root of the square of that number. That is, $|x| = +\sqrt{x^2}$. Use of this definition to solve $|x + 3| < 5$ is shown here:

$$|x + 3| < 5$$
$$+\sqrt{(x + 3)^2} < 5$$
$$(x + 3)^2 < 25$$
$$x^2 + 6x + 9 < 25$$
$$x^2 + 6x - 16 < 0$$
$$(x + 8)(x - 2) < 0$$

If the product of 2 factors is less than zero, *one and only one* factor is less than 0. If $(x+8) < 0$, then $x<-8$, but that would also make $(x-2) <0$ because $-8-2<0$. Consider $(x-2)<0$ or $x<2$. Since both factors can not be less than 0, we know that

$x > -8$ and $x < 2$

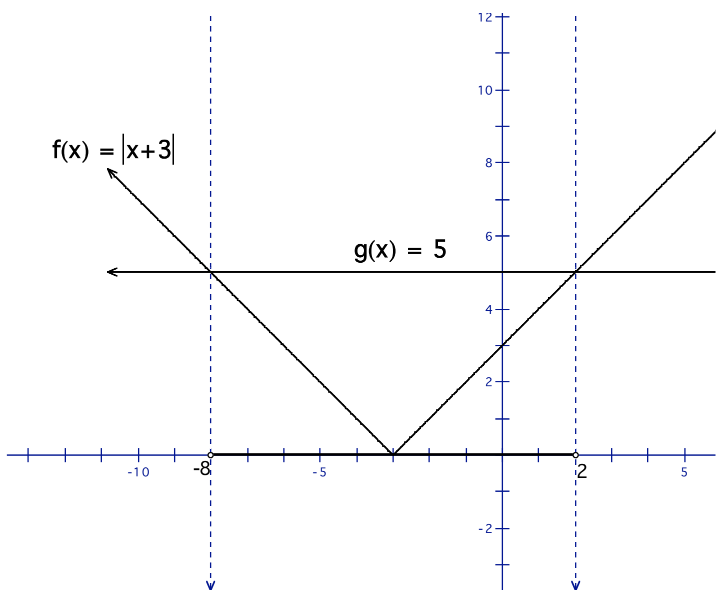
$-8 < x < 2$

Mathematical Focus 5

The solutions to $|x + 3| < 5$ can be understood using functions, and can be represented graphically.

The solution of an equation can be found by graphing the function related to the left member of the equation and the function related to the right member of the equation and finding the x-value of intersection point(s) of the graphs. Similarly, graphs of related functions can be used to determine the solution of an inequality. To solve the inequality $f(x) < g(x)$, find the values of x for which the graph of f indicates greater output values than the graph of g (one might think of this as the graph of f is "above" the graph of g).

Consider the functions $f(x) = |x + 3|$ and $g(x) = 5$. The solution to will be all values in the domain for which $f(x)$ is less than $g(x)$. The two functions are graphed in the first figure below. The solution to $|x + 3| < 5$ can be seen by determining for which x-values the graph of $f(x) = |x + 3|$ is below the graph of $g(x) = 5$. As shown below, $f(x)$ is less than $g(x)$ exactly when $-8 < x < 2$.



Now consider the inequality $x + 3 < 5$. Its solution will be all values in the domain for which the graph of $f(x) = x + 3$ is below the graph of $g(x) = 5$. As can be seen from the graph in the figure below, $f(x)$ is less than $g(x)$ exactly when $x < 2$. Comparing the two graphs, it is clear that the two solution sets are not equivalent. This graph highlights the fact that x<-8 is a solution for x+3<5 but it is not a solution for $|x + 3| < 5$.